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1992 J. Phys. A: Math. Gen. 25 6113

(http://iopscience.iop.org/0305-4470/25/22/032)

J. Phys. A: Math. Gen. 25 (1992) 6113–6121. Printed in the UK

Berry's phase in a moving boundary problem

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Received 3 February 1992, in final form 16 June 1992

Abstract. For a system in a well with a moving boundary we show that the s system memorizes the history of motion of the boundary through a non-integrable phase factor, Berry's phase. We first treat the case that the transformed Hamiltonian has a constant frequency. Secondly, in the case of time-dependent frequency we discuss non-adiabatic and non-cyclic cases.

1. Introduction

When a boundary of a system is moving, does the state of the system memorize the history of its motion? The answer is yes. In this paper, we will justify this by use of the non-integrable (history-dependent) property of Berry's phase [1].

There had been a prejudice such that a phase factor in a wavefunction which is a solution of the time-dependent Schrödinger equation can always be removed by a reparametrization of the phase with no change in the physics of the system. However, Berry [1] firstly clarified that there is a phase which cannot be removed by any reparametrization of the phase of the instantaneous eigenstate of the Hamiltonian whenever the evolution of the parameters for the Hamiltonian is cyclic. He used the adiabatic approximation in his discussions. Berry's phase has given a unique basic physics to a number of phenomena appearing in diverse fields in physics [2]. Wilczek and Zee [3], Aharonov and Anandan [4] and Samuel and Bhandari [5] have generalized the theory of Berry's phase to degenerate, non-adiabatic and non-cyclic cases respectively. It was also found by Hannay and Berry [6] that there is a classical limit of Berry's phase which is called Hannay's angle.

We consider the one-dimensional Schrödinger equation in cartesian coordinates. The solution is limited to the inner part of a potential well, constituting two infinite walls. The left boundary is chosen as the origin of position while the right boundary is allowed to undergo an arbitrary time-dependent shift. At any instant τ, the position of the right boundary is $L\bar{X}(\tau/T)$ where L and T are unit length and unit time, and $\bar{X}$ is an arbitrary dimensionless function. Assuming $\bar{X}(0) = 1$, L represents the position of the moving wall at the origin of time. We also assume the constant value of the potential inside the well and choose it as the origin of energies. Under these assumptions the wavefunction must satisfy

$$\begin{aligned} &\mathrm{i}\hbar\frac{\partial\hat{\psi}}{\partial\tau} = -\frac{\hbar^2}{2}\frac{\partial^2\hat{\psi}}{\partial x^2} && 0 \leqslant x \leqslant L\bar{X}(\tau/T) \\ &\hat{\psi}(\tau, x) = 0 && \forall\tau,\ x < 0 \text{ or } x > L\bar{X}(\tau/T). \end{aligned} \tag{1}$$

Since we are interested in evolution problems we must add the initial condition $\hat{\psi}(\tau = 0, x) = \hat{\psi}_0(x)$. Let us define a new rescaled space coordinate $q = x/\bar{X}(\tau/T)$, a new time variable $t = \int_0^\tau \mathrm{d}\sigma/\bar{X}^2(\sigma/T)$, and a new wavefunction

$$\psi(t,q) = \sqrt{X}\,\bar{\psi}(t,q)\exp\left(-\mathrm{i}\frac{1}{2\hbar}\frac{\dot{X}}{X}\frac{q^2}{T}\right) \tag{2}$$

where $\bar{\psi}(t,q) = \hat{\psi}(\tau,x)$, $X(t/T) = \bar{X}(\tau/T)$ and the dot denotes differentiation with respect to t/T. Then the new wavefunction satisfies [7]

$$\begin{aligned} &\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial q^2} + \frac{1}{2}q^2\Omega^2\psi \qquad && 0 \leqslant q \leqslant L \\ &\psi(t,q) = 0 && \forall t,\ q \leqslant 0 \ \text{or}\ q \geqslant L \end{aligned} \tag{3}$$

with

$$\Omega^2(t/T) = [\ddot{X}/X - 2(\dot{X}/X)^2]/T^2 = \bar{X}^3\bar{X}''/T^2 \tag{4}$$

where the prime denotes differentiation with respect to τ/T. The new initial condition associated with the problem is $\psi_0(q) = \psi(t=0,q) = \bar{\psi}_0(q)\exp[-\mathrm{i}q^2\dot{X}(0)/2\hbar T]$. Whether the frequency given in (4) is time-dependent or time-independent depends on the motion of the right boundary, i.e. the function form of $\bar{X}(\tau/T)$.

2. Constant frequency case

Suppose that the right boundary is moving such that

$$\bar{X}(\tau/T) = [1 + (a/T)\tau]^{1/2}. \tag{5}$$

Then the new rescaled coordinate q, the new time variable t and the frequency Ω^2 are given as the following: $q = x[1+(a/T)\tau]^{1/2}$, $t = (T/a)\ln[1+(a/T)\tau]$ and $\Omega^2 = -(a/2T)^2$. Thus our original problem with a moving boundary reduces to that of the time-independent harmonic oscillator. In order to obtain Berry's phase in this problem we resort to the Gaussian wavepacket approximation.

The Schrödinger equation is equivalent to $\delta I = 0$ where the action is given by $I = \int[\langle\psi|\mathrm{i}\hbar\partial_t|\psi\rangle - \langle\psi|\hat{H}|\psi\rangle]\,\mathrm{d}t$. In order to take the Gaussian wavepacket approximation we consider the family of normalized coherent states

$$\psi(q,t) = (\pi\hbar)^{-n/4}\exp\{[-\tfrac{1}{2}|q - Q(t)|^2 + \mathrm{i}P(t)\cdot(q - Q(t)) + \mathrm{i}\gamma(t)]/\hbar\} \tag{6}$$

where n is the dimension of q, γ is the phase, $\boldsymbol{Q}$ and $\boldsymbol{P}$ specify a point in an ordinary $2n$-dimensional classical phase space. Thus there are $2n+1$ parameters $\boldsymbol{G} = (\gamma, \boldsymbol{Q}, \boldsymbol{P})$. Parameters which give the true solution of the time-dependent Schrödinger equation satisfy the following equations: $\dot{Q}_i = \partial H(\boldsymbol{Q},\boldsymbol{P})/\partial P_i$, $-\dot{P}_i = \partial H(\boldsymbol{Q},\boldsymbol{P})/\partial Q_i$ and $\dot{\gamma} = \boldsymbol{P}\cdot\dot{\boldsymbol{Q}} - H(\boldsymbol{Q},\boldsymbol{P})$ where $H(\boldsymbol{Q},\boldsymbol{P})$ is the classical Hamiltonian. Now we renormalize the energy into zero thus the Berry connection $A(t)$ is given by

$$A(t) = \langle\psi(t)|\mathrm{i}\partial/\partial t|\psi(t)\rangle = \dot{\gamma} - \boldsymbol{P}\cdot\dot{\boldsymbol{Q}}. \tag{7}$$

Then, as is expected [1], the Berry connection $A(t)$ is zero along the actual evolution curve in the energy renormalized Hilbert space. Thus the Berry phase arises from the line integral of the Berry connection along a geodesic connecting two end points of the actual evolution curve. This is called the geodesic rule [5]. However, for a cyclic evolution the projection of the actual evolution curve into the projective Hilbert space is a closed one, so there is no contribution from the projection of the geodesic. Now we can define a vector $\tilde{\psi}(q,t)$ in the projective Hilbert space such that

$$\tilde{\psi}(\boldsymbol{q},t) = (\pi\hbar)^{-n/4}\exp\{[-\tfrac{1}{2}|\boldsymbol{q}-\boldsymbol{Q}(t)|^2 + \mathrm{i}\boldsymbol{P}(t)\cdot(\boldsymbol{q}-\boldsymbol{Q}(t))]/\hbar\}. \tag{8}$$

Since there is one-to-one correspondence between the state $\tilde{\psi}(\boldsymbol{q};t)$ and $2n$ parameters $\tilde{\boldsymbol{G}} = (\boldsymbol{Q},\boldsymbol{P})$, the time derivative $\partial/\partial t$ can be replaced by $\nabla_{\tilde{\boldsymbol{G}}}\cdot\dot{\tilde{\boldsymbol{G}}}$. Then the Berry connection $\tilde{A}(t)$ in the projective Hilbert space is given by [8]

$$\tilde{A}(t) = \langle\tilde{\psi}|\mathrm{i}\nabla_{\boldsymbol{Q}}|\tilde{\psi}\rangle\cdot\dot{\boldsymbol{Q}} + \langle\tilde{\psi}|\mathrm{i}\nabla_{\boldsymbol{P}}|\tilde{\psi}\rangle\cdot\dot{\boldsymbol{P}} = \boldsymbol{P}\cdot\dot{\boldsymbol{Q}}. \tag{9}$$

Note that $A(t)$ is identically zero but $\tilde{A}(t)$ is not always zero. By use of the Gaussian wavepacket approximation we can easily see that the Aharonov–Bohm phase and the Collela, Overhauser and Werner (COW) phase are examples of Berry's phase [9].

The classical Hamiltonian for the simple harmonic oscillator with $\Omega^2 = -(a/2T)^2$ is given by

$$H(Q,P) = \tfrac{1}{2}P^2 - \tfrac{1}{2}a/2T)^2Q^2 \tag{10}$$

where Q and P are parameters for the Gaussian wavepacket. The parameters Q and P satisfying the following equations: $\dot{Q}(t) = \partial H(Q,P)/\partial P = P(t)$ and $\dot{P}(t) = -\partial H(Q,P)/\partial Q = (a/2T)^2Q(t)$ and boundary conditions $Q(0) = 0$ and $P(0) = \dot{Q}(0) = v$ can be written such that

$$Q(t) = \begin{cases} \dfrac{2T}{a}v\sinh\left(\dfrac{a}{2T}(t-2nT_\mathrm{b})\right) & 2nT_b \leqslant t \leqslant (2n+1)T_\mathrm{b} \\ -\dfrac{2T}{a}v\cosh\left(\dfrac{a}{2T}T_\mathrm{b}\right)\sinh\left(\dfrac{a}{2T}(t-(2n+1)T_\mathrm{b})\right) & \\ +L\cosh\left(\dfrac{a}{2T}(t-(2n+1)T_\mathrm{b})\right) & (2n+1)T_\mathrm{b} \leqslant t \leqslant 2(n+1)T_\mathrm{b} \end{cases} \tag{11}$$

$$P(t) = \begin{cases} v\cosh\left(\dfrac{a}{2T}(t-2nT_\mathrm{b})\right) & 2nT_\mathrm{b} \leqslant t \leqslant (2n+1)T_\mathrm{b} \\ -v\cosh\left(\dfrac{a}{2T}T_\mathrm{b}\right)\cosh(\dfrac{a}{2T}(t-(2n+1)T_\mathrm{b})\Big) & \\ +\dfrac{aL}{2T}\sinh\left(\dfrac{a}{2T}(t-(2n+1)T_\mathrm{b})\right) & (2n+1)T_\mathrm{b} \leqslant t \leqslant 2(n+1)T_\mathrm{b} \end{cases} \tag{12}$$

where $n = 0,1,2,\ldots$ and T_b is the time at which the wavepacket first bounces off the right wall, i.e. $Q(T_\mathrm{b}) = L$. Although the wavefunction is confined to a potential well with infinite walls the form of Gaussian wavepacket we choose has non-vanishing quantum tails to infinity. We have used a naive approximation which is correct only at the classical level. The Berry phase that the state will attain after N-times round-trips, γ_N, is obtained as

$$\begin{aligned}\gamma_N &= N\gamma_1 = 2N\int_0^{T_b} P(t)\dot{Q}(t)\,\mathrm{d}t \\ &= 2Nv^2\left[\frac{T_b}{2} + \frac{T}{2a}\sinh(\frac{a}{T}T_b)\right].\end{aligned} \tag{13}$$

The solution of the original problem given in (1), $\hat{\psi}(\tau, x)$ can be obtained from the Gaussian wavepacket solution $\psi(t, q)$ by use of (2)

$$\begin{aligned}\hat{\psi}(\tau, x) = \left[\pi\hbar(1+\frac{a}{T}\tau)\right]^{-1/4} \exp\Bigg[&-\frac{1}{2\hbar}(1+\frac{a}{T}\tau)^{-1}\left\{x - \bar{Q}(\tau)\left(1+\frac{a}{T}\tau\right)^{1/2}\right\}^2 \\ &+\mathrm{i}\frac{1}{\hbar}\left(1+\frac{a}{T}\tau\right)^{-1/2}\bar{P}(\tau)\left\{x - \bar{Q}(\tau)\left(1+\frac{a}{T}\tau\right)^{1/2}\right\} \\ &+\mathrm{i}\frac{1}{2\hbar}\frac{a}{2T^2}(1+\frac{a}{T}\tau)^{-1}x^2\Bigg]\end{aligned} \tag{14}$$

where $Q(t) = \bar{Q}(\tau)$ and $P(t) = \bar{P}(\tau)$. If $T \gg \sqrt{a}$, the last term in the exponent in (14) vanishes and therefore $\hat{\psi}(\tau, x)$ is also a normalized Gaussian wavepacket.

3. Time-dependent frequency case

For the generalized harmonic oscillator, the Hamiltonian is given by $\frac{1}{2}X(t)q^2 + 2Y(t)qp + Z(t)p^2)$. If $X(t)$, $Y(t)$ and $Z(t)$ are slowly varying parameters, i.e. the evolution is adiabatic, then the Berry connection for an energy eigenstate, $\tilde{A}_n(t)$, is given by $-\frac{1}{2}(n+\frac{1}{2}(\dot{Z}Y - \dot{Y}Z)/(Z\sqrt{XZ-Y^2})$ [6]. Hence the Berry connection for the system described by (3) vanishes if the evolution of the system is adiabatic since $Y(t) = 0$. Thus we must treat a non-adiabatic case [4] in order to show the appearance of Berry's phase in a time-dependent harmonic oscillator problem.

For the non-adiabatic case, we may find the geometric phase by use of an instantaneous eigenstate $|n(t)\rangle$ of the Lewis invariant $\hat{I}(t)$ [10] such as $\int_0^T \langle n(t)|\mathrm{i}\partial/\partial t|n(t)\rangle\,\mathrm{d}t$. In the adiabatic approximation, there is one-to-one correspondence between the set of parameters $\{h_i(t)\}$ and the instantaneous eigenstates of the Hamiltonian $\hat{H}(t)$. Similarly, in general, there is one-to-one correspondence between the set of generalized parameters $\{\alpha_i(t)\}$ (not $\{h_i(t)\}$) [11] and the instantaneous eigenstates of the invariant $\hat{I}(t)$. Hence the operator $\partial/\partial t$ may be replaced by the form $\nabla_\alpha \cdot (\mathrm{d}\alpha/\mathrm{d}t)$ and consequently we obtain an analogous solid-angle law or an area law for the geometric phase $\gamma_n(C)$ in the non-adiabatic case [11]

$$\gamma_n(C) = \int_0^T \langle n(t)|\mathrm{i}\partial/\partial t|n(t)\rangle\,\mathrm{d}t = \int_{\alpha(0)}^{\alpha(T)} \langle n(\alpha)|\mathrm{i}\nabla_\alpha|n(\alpha)\rangle \cdot \mathrm{d}\alpha \tag{15}$$

where $\alpha(t)$ denotes $\{\alpha_i(t)\}$.

For the time-dependent harmonic oscillator described by (3), the Hamiltonian $\hat{H}(t)$ is given by $\frac{1}{2}p^2 + \frac{1}{2}\Omega^2(t)q^2$. Then the invariant $\hat{I}(t)$ is given by

$$\hat{I}(t) = \frac{1}{2}\left[p^2 - 2\frac{\dot{\rho}}{\rho}qp + \left(\frac{1}{\rho^4} + \frac{\dot{\rho}^2}{\rho^2}\right)q^2\right] \tag{16}$$

where $\rho(t)$ satisfies the following non-autonomous nonlinear differential equation $\ddot{\rho} + \Omega^2(t)\rho - (1/\rho^3) = 0$. We assume that the state of the system is initially in an eigenstate of the Hamiltonian $\hat{H}(0)$. In other words, we set $\hat{I}(0) = \hat{H}(0)$, i.e. $\rho(0) = 1/\sqrt{\Omega(0)}$ and $\dot{\rho}(0) = 0$. Let us set $X(t) = (1/\rho^4) + (\dot{\rho}^2/\rho^2)$, $Y(t) = -(\dot{\rho}/\rho)$ and $Z(t) = 1$; then the Berry connection in the projective Hilbert space is written such that

$$\tilde{A}_n(t) = -\frac{1}{2}(n + \tfrac{1}{2})\frac{\dot{Z}Y - \dot{Y}Z}{Z(XZ - Y^2)^{1/2}} = -\frac{1}{2}(n + \tfrac{1}{2})(\rho\ddot{\rho} - \dot{\rho}^2) \tag{17}$$

where $X(t)$, $Y(t)$ and $Z(t)$ are parameters in the generalized parameter space. Note that we must choose special quantum numbers n satisfying $n(t,0) = n(t,L) = 0$, i.e. $\psi(t,0) = \psi(t,L) = 0$, which are the boundary conditions given in (3).

As a specific example, we consider a case in which the right boundary is oscillating with frequency ω and amplitude $\cosh(a/\omega)$, i.e. $\tilde{X}(\tau)$ is given by

$$\tilde{X}(\tau) = \begin{cases} \cosh[a(\tau - 2n/\omega)] & 2n/\omega \leqslant \tau \leqslant (2n+1)/\omega \\ \cosh[a(2(n+1)/\omega - \tau)] & (2n+1)/\omega \leqslant \tau \leqslant 2(n+1)/\omega \end{cases} \tag{18}$$

where $n = 0, 1, 2, \ldots$. Then the new time variable t is given by

$$t = \begin{cases} (2n/a)\tanh(a/\omega) + (1/a)\tanh[a(\tau - 2n/\omega)] \\ \qquad\qquad 2n/\omega \leqslant \tau \leqslant (2n+1)/\omega \\ (2(n+1)/a)\tanh(a/\omega) + (1/a)\tanh[a(2(n+1)/\omega - \tau)] \\ \qquad\qquad (2n+1)/\omega \leqslant \tau \leqslant 2(n+1)/\omega \end{cases} \tag{19}$$

and the frequency Ω^2 is given by

$$\Omega^2(t) = \begin{cases} a^2\cosh^4\{\tanh^{-1}[at - 2n\tanh(a/w)]\} \\ \qquad\qquad 2n/\omega \leqslant \tau \leqslant (2n+1)/\omega \\ a^2\cosh^4\{\tanh^{-1}[2(n+1)\tanh(a/w) - at]\} \\ \qquad\qquad (2n+1)/\omega \leqslant \tau \leqslant 2(n+1)/\omega. \end{cases} \tag{20}$$

For our setting, $\tilde{X}(0) = 1$. The fact that $\tilde{X}'(0) = 0$ guarantees the invariance of the initial condition, i.e. $\psi_0(q) = \tilde{\psi}_0(q) = \hat{\psi}_0(x)$. If we solve the nonlinear equation $\ddot{\rho} + \Omega^2(t)\rho - (1/\rho^3) = 0$ with the frequency $\Omega^2(t)$ given in (20) and the initial conditions $\dot{\rho}(0) = 0$ and $\rho(0) = 1/\sqrt{\Omega(0)}$ by use of a numerical method, we can see that the trajectory in the ρ–$\dot{\rho}$ plane is an open curve, in general. In other words, there is no time T at which $\hat{I}(T) = \hat{I}(0)$. Thus the evolution is non-cyclic. We will discuss non-cyclic cases as the next step for showing the appearance of Berry's phase for a time-dependent harmonic oscillator.

4. Non-cyclic cases

If we denote $|\phi_n(t)\rangle$ as a solution of the time-dependent Schrödinger equation $(\mathrm{i}\hbar\partial/\partial t - \hat{H}(t))|\phi_n(t)\rangle = 0$ with the boundary condition $|\phi_n(0)\rangle = |n(0)\rangle$ where $|n(0)\rangle$ is an instantaneous eigenstate of $\hat{H}(0)$, then $|\phi_n(T)\rangle$ is given by [10]

$$|\phi_n(T)\rangle = |n(T)\rangle \exp\left[\mathrm{i}\int_0^T \langle n(t)|\mathrm{i}\partial/\partial t|n(t)\rangle\,\mathrm{d}t - \frac{\mathrm{i}}{\hbar}\int_0^T \langle n(t)|\hat{H}(t)|n(t)\rangle\,\mathrm{d}t\right] \quad (21)$$

where $|n(t)\rangle$ is the instantaneous eigenstate of the Lewis invariant $\hat{I}(t)$ which satisfies $\hat{I}(0) = \hat{H}(0)$. If we renormalize the energy so as to be zero at every time then the evolution of the state can be regarded as a series of dense measurements, which may be written as

$$|\phi_n(T)\rangle = |n(T)\rangle \lim_{\epsilon\to 0}\prod_{k=1}^{N}\langle n(k)|n(k-1)\rangle = |n(T)\rangle \exp\left[\mathrm{i}\int_0^T \tilde{A}_n(t)\,\mathrm{d}t\right] \quad (22)$$

where $\epsilon = T/N$ and $\tilde{A}_n(t) \equiv \langle n(t)|\mathrm{i}\partial/\partial t|n(t)\rangle$. Note that if one changes $|n(t)\rangle \to \exp[\mathrm{i}\chi(t)]|n(t)\rangle$ then $|\phi_n(T)\rangle \to \exp[\mathrm{i}\chi(0)]|\phi_n(T)\rangle$, that is, $|\phi_n(T)\rangle$ is independent to the phase arbitrariness of $|n(t)\rangle$ for $0 < t \leqslant T$ [5]. If $|n(T)\rangle \neq |n(0)\rangle$ then the evolution is called a non-cyclic one. We will discuss that, for a non-cyclic evolution, there are two ways to measure Berry's phase.

The projective Hilbert space is one which is spanned by the instantaneous eigenstates $\{|n(t)\rangle\}$ of the invariant $\hat{I}(t)$. Let $|R\rangle$ be a state in the projective Hilbert space which is not equal to the initial state $|n(0)\rangle$ (see figure 1). Let us obtain the phase difference between $|R\rangle$ and $|\phi_n(T)\rangle$. In order to do this we consider the inner product of these two states

$$\begin{aligned}\langle R|\phi_n(T)\rangle &= \langle R|n(T)\rangle \exp\left[\mathrm{i}\int_0^T \tilde{A}_n(t)\,\mathrm{d}t\right] \\ &= \|\langle R|n(T)\rangle\| \exp\left[\mathrm{i}\int_{\tilde{g}} \tilde{A}_s\,\mathrm{d}s\right]\exp[\mathrm{i}\beta_f].\end{aligned} \quad (23)$$

where $\tilde{g}$ is a geodesic connecting $|R\rangle$ and $|n(T)\rangle$ (see figure 1) and $\beta_f \equiv \int_0^T \tilde{A}_n(t)\,\mathrm{d}t$. We applied the geodesic rule [5] in the calculation of $\langle R|n(T)\rangle$. Thus the phase difference β_i^R between $|R\rangle$ and $|\phi_n(T)\rangle$ is given by $\beta_i^R = \int_{\tilde{E}+\tilde{g}} \tilde{A}_s\,\mathrm{d}s$ where $\tilde{E}$ is the projection of the actual evolution curve E, which is shown in figure 1. Note that under the local phase reparametrization, the phase β_i^R is transformed such that $\beta_i^R \to [\chi(0) - \chi(R)] + \beta_i^R$, that is, β_i^R is independent of the phase arbitrariness along the curve $\tilde{E} + \tilde{g}$ except at the end point of the curve. Now we consider another evolution curve E' and its projection $\tilde{E}'$ (corresponding to another Hamiltonian), in which $\tilde{E}'$ has the same end points as those of $\tilde{E}$ (see figure 1). Then the phase difference $\beta_i^{R'}$ between $|R\rangle$ and $|\phi_n(T)\rangle'$ is given by $\beta_i^{R'} = \int_{\tilde{E}'+\tilde{g}} \tilde{A}_s\,\mathrm{d}s$. Under the local phase reparametrization, the phase $\beta_i^{R'}$ is transformed such that $\beta_i^{R'} \to [\chi(0) - \chi(R)] + \beta_i^{R'}$. Then the difference of the phases β_i^R and $\beta_i^{R'}$

$$\beta_i^{R'} - \beta_i^R = \oint_{\tilde{E}'-\tilde{E}} \tilde{A}_s\,\mathrm{d}s = \beta_f' - \beta_f \quad (24)$$

is invariant under the local phase reparametrization, and is therefore an observable. Note that the difference of the phases, $\beta_i^{R'} - \beta_i^R$ is irrelevant to the reference state $|R\rangle$. An experiment has been carried out in order to measure this phase β_f by tracing the neutron spin on its passage through a magnetic field in the $\hat{x}$–$\hat{y}$ plane at various frequencies [12]. For the time-dependent harmonic oscillator, the phase $\beta_{f,n}$ for the quantum state $|n(t)\rangle$ is given by

$$\beta_{f,n} = -\frac{1}{2}(n + \tfrac{1}{2}) \int_0^T (\rho\ddot{\rho} - \dot{\rho}^2)\,\mathrm{d}t \tag{25}$$

by use of $\tilde{A}_n(t)$ which is given in (17).

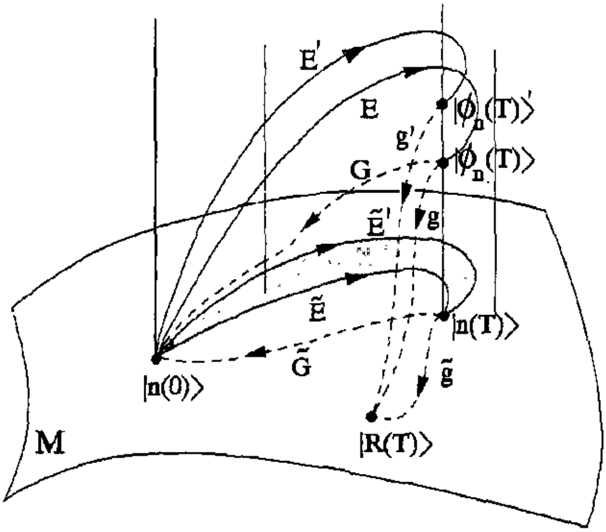

Figure 1. Schematic diagram of the Hilbert space. M denotes the projective Hilbert space. G, $\tilde{G}$, g, g' and $\tilde{g}$ are geodesic E and E' are actual evolution curves for two different Hamiltonians and $\tilde{E}$ and $\tilde{E}'$ are projections of E and E' into M. An observable $\oint_{\tilde{E}-\tilde{E}'} \tilde{A}_s\,\mathrm{d}s$ is given by the shaded area. Since $E + G$ as well as $\tilde{E} + \tilde{G}$ is closed, the phase $\oint_{\tilde{E}+\tilde{G}} \tilde{A}_s\,\mathrm{d}s$ is also an observable.

Let us consider the case where the reference state $|R\rangle$ is equal to $|n(0)\rangle$. Then the phase difference β_i^0 between $|n(0)\rangle$ and $|\phi_n(T)\rangle$ is given by

$$\beta_i^0 = \oint_{\tilde{E}+\tilde{G}} \tilde{A}_s\,\mathrm{d}s \tag{26}$$

where $\tilde{G}$ is a geodesic connecting $|n(0)\rangle$ and $|n(T)\rangle$ as in figure 1. The closure of the curve $\tilde{E}+\tilde{G}$ guarantees the invariance of β_i^0 under the local phase reparametrization along the curve. Therefore the phase β_i^0 itself is an observable and we may observe the absolute value of β_i^0 [5]. An experiment to observe the phase β_i^0 by measuring the rotation of the plane of polarization of a linearly polarized beam travelling along a uniformly wound half-turn single-mode optical fibre is described in [13]. In order to obtain the phase $\beta_{i,n}^0$ for an eigenstate $|n(t)\rangle$ of the invariant $\hat{I}(t)$ for the time-dependent harmonic oscillator problem we need to obtain geodesics for this problem. The invariant $\hat{I}(t)$ in (16) can be written as $\hat{I}(t) = \sum_{i=1}^{3} R^i(t)T_i$ where $R^1 = 1 - 1/\rho^4 - \dot{\rho}^2/\rho^2, R^2 = -2\dot{\rho}/\rho$ and $R^3 = 1 + 1/\rho^4 + \dot{\rho}^2/\rho^2$, and T_1 $(= \frac{1}{4}(q^2 - p^2))$, $T_2 (= \frac{1}{4}(pq + qp))$ and T_3 $(= \frac{1}{4}(q^2 + p^2))$ are generators of the group SO(2,1) with commutation relations $[T_i, T_j] = \mathrm{i}\epsilon_{ijk}T^k (g_{ij} = \mathrm{diag}(-1, -1, 1))$. It can be easily shown that $(R^3)^2 - (R^2)^2 - (R^1)^2 = 4(XZ - Y^2) = \Delta^2$. Since Berry's phase is indifferent to the magnitude of Δ and $R^3 > 0$, we choose the upper sheet of the unit hyperboloid corresponding to the group SO(2,1)/SO(2) as the generalized parameter space [14]. The vector $\boldsymbol{R}$ can be parametrized as $\boldsymbol{R} = (\cos\theta \sinh\beta, \sin\theta \sinh\beta, \cosh\beta)$ $(0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \beta < \infty)$. Then the general geodesic equation is given by

$$\theta = -\mathrm{i}\frac{E}{|E|}\frac{1}{2}\ln\frac{\sqrt{1 - E^{-2}\sinh^2\beta} - \cosh\beta}{\sqrt{1 - E^{-2}\sinh^2\beta} + \cosh\beta} + D \equiv f(\beta, E) + D. \tag{27}$$

From the values $(\rho(0), \dot{\rho}(0))$ and $(\rho(T), \dot{\rho}(T))$, coordinates (θ_1, β_1) and (θ_2, β_2) which represent the starting and end points of the evolution curve in the unit hyperboloid are easily obtained. Then we may determine E and D by solving two equations $\theta_1 = f(\beta_1, E) + D$ and $\theta_2 = f(\beta_2, E) + D$, i.e. we may find the geodesic connecting two end points of the evolution curve. Then the phase $\beta_{i,n}^0$ for the quantum state $|n(t)\rangle$ is given by

$$\oint_{\text{evo}-\text{geo}} \bar{A}_s\,\mathrm{d}s = -\frac{1}{2}(n + \tfrac{1}{2})\left[\int_0^T (\rho\ddot{\rho} - \dot{\rho}^2)\,\mathrm{d}t - \int_{\beta_i}^{\beta_f} \frac{E\cosh\beta}{\sinh\beta\sqrt{\sinh^2\beta - E^2}}\,\mathrm{d}\beta\right]. \tag{28}$$

5. Discussions

We have shown that Berry's phase appears in the state of a particle moving under a constant potential in a well with a moving wall. Motivations for our problem are as follows: one is academic and the others are somewhat physical. An academic motivation is that our problem is an example in which the non-adiabatic geometric phase plays a crucial role. Since the Hamiltonian for our problem in the transformed space and time, which is given by (3), is that for a time-dependent harmonic oscillator, Berry's phase does not appear if the evolution is adiabatic. Only when the evolution is non-adiabatic can the geometric phase have a non-trivial value. In addition, the system never comes back to the starting state in general. Thus our problem is an example of the non-adiabatic and non-cyclic geometric phase. A physical motivation for our problem is that we explicitly show that the state of the system memorizes the history of motion of the boundary through a non-integrable (history-dependent) phase factor, Berry's phase. Another physical motivation to study the time-dependent harmonic oscillator type Hamiltonian in (3) is that for the problem of squeezed-light generation a resonator containing some medium with a time-dependent dielectric permeability function is usually considered and, in the one-dimensional case, the vector-potential of the electromagnetic field satisfies the equation of an oscillator with a time-dependent frequency [15]. Other relevant physical problems which are associated with the one-dimensional time-dependent harmonic oscillators are listed in the textbook [16].

Now we discuss why the exhibited Berry's phases are non-trivial. In the case of the Gaussian wavepacket approximation, the parameters for the projective Hilbert space are Q and P (see (9)). For the constant frequency, any trajectory in Q–P space must be cyclic since the Hamiltonian is a time-independent one-dimensional harmonic oscillator. Thus Berry's phase appearing in our problem is non-trivial. When we use the Lewis invariant method, there exist two ways to observe Berry's phases appearing in non-cyclic evolutions. The phases in (24) and (26) are the observable phases, which are invariant under the local phase reparametrization, and they have been observed experimentally [5, 12].

Acknowledgments

We are grateful for useful discussions with Professors Hyun Kyu Lee and Y S Myung. This research was supported in part by the Basic Science Research Institute Program, Ministry of Education, 1991, BSRI 91-103, and by the Korea Science and Engineering Foundation through the Center for Theoretical Physics, Seoul National University.

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